

CARLESON MEASURES AND MULTIPLIERS OF DIRICHLET-TYPE SPACES

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ABSTRACT. A function ρ from $[0,1]$ onto itself is a Dirichlet weight if it is increasing, $\rho'' \leq 0$ and $\lim_{x \rightarrow 0^+} x/\rho(x) = 0$. The corresponding Dirichlet-type space, D_ρ , consists of those bounded holomorphic functions on $U = \{z \in \mathbf{C} : |z| < 1\}$ such that $|f'(z)|^2 \rho(1 - |z|)$ is integrable with respect to Lebesgue measure on U . We characterize in terms of a Carleson-type maximal operator the functions in the set of pointwise multipliers of D_ρ , $M(D_\rho) = \{g: U \rightarrow \mathbf{C} : gf \in D_\rho, \forall f \in D_\rho\}$.

I. Introduction. Let $H(U)$ denote the set of functions holomorphic in the open unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$. A function ρ mapping $[0, 1]$ onto itself is called a Dirichlet weight if it is increasing, concave in the strong sense that $\rho'' \leq 0$ and $\lim_{x \rightarrow 0^+} x/\rho(x) = 0$. Given such a weight, the Dirichlet-type space D_ρ consists of those $f \in H(U)$ for which the norm

$$(1.1) \quad \|f\|_\rho = |f(0)| + \left[\iint_U |f'(z)|^2 \rho(1 - |z|) dz \right]^{1/2}$$

is finite; here dz denotes Lebesgue measure on U . The main purpose of this paper is to characterize the set $M(D_\rho)$ of pointwise multipliers of D_ρ , where

$$M(D_\rho) = \{g: U \rightarrow \mathbf{C} : f \in D_\rho \text{ for all } f \in D_\rho\}.$$

Previous results concerning pointwise multipliers of these spaces dealt with the special case

$$D_\alpha = \left\{ f \in H(U), f = \sum_{n=0}^{\infty} a_n z^n : \|f\|_\alpha = \left[\sum_{n=0}^{\infty} (1 + n^2)^\alpha |a_n|^2 \right]^{1/2} < \infty \right\},$$

$-\infty < \alpha < \infty$; in case $\alpha = 1/2$ this is the classical Dirichlet space of functions in $H(U)$ whose derivatives are square-integrable on U . Taylor [10] described $M(D_\alpha)$ when α lies outside $(0, 1/2]$: for $\alpha \leq 0$, $M(D_\alpha) = H^\infty(U) = H(U) \cap L^\infty(U)$ while for $\alpha > 1/2$, $M(D_\alpha) = D_\alpha$ (so that D_α is an algebra). D. Stegenga [8] showed that for $\rho_\alpha(r) = r^{1-2\alpha}$, $\alpha < 1$,

$$(1.2) \quad \|f\|_\alpha \approx |f(0)| + \left[\iint_U |f'(z)|^2 \rho_\alpha(1 - |z|) dz \right]^{1/2};$$

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that is, the two terms in (1.2) are equivalent in the sense that each is no larger than a constant multiple of the other, the constants being independent of f . (We observe $[0, 1/2]$ is precisely the range of α for which ρ_α is concave.) Stegenga proved $g \in M(D_\alpha)$, $\alpha \in (0, 1/2]$, if and only if $g \in H^\infty(U)$ and

$$(1.3) \quad \int_{\bigcup_j S(I_j)} |g'(z)|^2 \rho_\alpha(1 - |z|) dz \leq C \text{Cap}_\alpha \left(\bigcup_j I_j \right)$$

for all finite collections of pairwise disjoint subarcs $\{I_j\}$ on the unit circle $T = \{z \in \mathbf{C} : |z| = 1\}$. Here $S(I)$ denotes the square $\{z = re^{i\theta} : e^{i\theta} \in I \text{ and } (1 - |I|)_+ < r < 1\}$, $|I|$ is the arclength of the subarc I , and $\text{Cap}_\alpha(E)$ denotes the Bessel capacity of order α of the set E .

Our characterization of $M(D_\rho)$ differs from (1.3) in that we replace the capacity by a Carleson-type maximal operator and so are able to test a certain inequality, (1.4), over subarcs of T rather than finite unions of subarcs. This maximal operator, M_ρ , associates to nonnegative h on U a function $M_\rho h$ on T by

$$(M_\rho h)(e^{i\theta}) = \sup [|I| \rho(|I|)]^{-1/2} \iint_{S(I)} h(z) dz,$$

the supremum being over all subarcs I of T containing $e^{i\theta}$. Adopting the usual notation, χ_E , for the characteristic function of the set E , we can now state

THEOREM A. *Suppose ρ is a Dirichlet weight. Then $g \in M(D_\rho)$ if and only if $g \in H^\infty(U)$ and there exists $C > 0$ such that*

$$(1.4) \quad \begin{aligned} & \int_I [M_\rho(\chi_{S(I)} |g'(\cdot)|^2 \rho(1 - |\cdot|))(e^{i\theta})]^2 d\theta \\ & \leq C \iint_{S(I)} |g'(z)|^2 \rho(1 - |z|) dz < \infty \end{aligned}$$

for all subarcs I of T .

We now outline in some detail the proof of Theorem A. There are two main steps. First, in §2, Plancherel's theorem is used to show the norm defined at $f(z) = \sum_{n=0}^\infty a_n z^n$ in D_ρ by

$$(1.5) \quad \|f\|_\rho^* = \left[|a_0|^2 + \sum_{n=1}^\infty n\rho\left(\frac{1}{n}\right) |a_n|^2 \right]^{1/2}$$

is equivalent to the one in (1.1). We next observe that, ρ being concave, $n\rho(1/n)$ is a nondecreasing function of n , so D_ρ must be contained in the Hardy space $H^2(T) = \{f \in L^2(T) : \hat{f}(n) = 0 \text{ for } n < 0\}$. Thus, defining the function K_ρ on T in terms of its Fourier coefficients

$$(1.6) \quad \hat{K}_\rho(n) = \begin{cases} 2, & n = 0, \\ [|n| \rho(1/|n|)]^{-1/2}, & n \neq 0, \end{cases}$$

we can identify D_ρ with the space S_ρ of Poisson integrals of K_ρ -potentials of functions in $H^2(T)$,

$$S_\rho = \{f(z) = (P_r * K_\rho * h)(e^{i\theta}) : z = re^{i\theta}, h \in H^2(T)\};$$

here, as usual,

$$(g * h)(e^{i\theta}) = \frac{1}{2\pi} \int_T g(e^{i(\theta-t)})h(e^{it}) dt$$

and $P_r(\theta) = (1 - r^2)/(1 - 2r \cos \theta + r^2)$. Indeed, if $f \in D_\rho$, then $f = P_r * K_\rho * h$ where

$$\hat{h}(n) = \begin{cases} a_0/2, & n = 0, \\ 0, & n < 0, \\ n\rho(1/n)a_n, & n > 0. \end{cases}$$

Second, by [5, Proposition 1.6/4], $M(D_\rho)$ is embedded in $H^\infty(U)$. As pointed out on p. 178 of [5], this fact, together with the product rule of differentiation shows $g \in M(D_\rho)$ if and only if $g \in H^\infty(U)$ and

$$\iint_U |f(z)|^2 |g'(z)|^2 \rho(1 - |z|) dz < \infty$$

for all $f \in D_\rho$. The identification of D_ρ with S_ρ described above then says $g \in M(D_\rho)$ if and only if $g \in H^\infty(U)$ and

$$(1.7) \quad \iint_U |(P_r * K_\rho * h)(e^{i\theta})|^2 |g'(re^{i\theta})|^2 \rho(1 - r) d(re^{i\theta}) < \infty$$

for all $h \in H^2(T)$. Taking complex conjugates, it is seen (1.7) would, in fact, hold for all $h \in L^2(T)$. By the closed graph theorem, this is equivalent to the existence of $C > 0$ such that

$$(1.8) \quad \iint_U |(P_r * K_\rho * h)(e^{i\theta})|^2 d\mu(re^{i\theta}) \leq C \int_T |h(e^{i\theta})|^2 d\theta$$

for all $h \in L^2(T)$, with $d\mu = |g'(z)|^2 \rho(1 - |z|) dz$. Any $\mu \in B(U)$, the class of positive Borel measures on U , which satisfies (1.8) will be called a Carleson measure on S_ρ . The proof of Theorem A is completed in §III by a characterization of such Carleson measures. This characterization is also used in §IV to give another approach to the ‘‘Féjer-Riesz inequality’’ of Nagel, Rudin and Shapiro [6].

II. Dirichlet-type spaces. In Theorem 2.2 below we require the following result, whose proof can be found, for example, in [2, p. 183].

LEMMA 2.1. *Suppose $\{c_n\}_{n=-\infty}^\infty$ is an even, nonnegative sequence on Z which is nonincreasing and convex on Z_+ with $\lim_{n \rightarrow \infty} c_n = 0$. Then the even function $K(e^{i\theta}) = \sum_{n=-\infty}^\infty c_n e^{-in\theta}$ is nonnegative and integrable on T .*

THEOREM 2.2. *Suppose ρ is a Dirichlet weight. Then the function K_ρ given through its Fourier coefficients in (1.6) is even, nonnegative and integrable on T . The mapping T_ρ defined in terms of K_ρ by*

$$(T_\rho h)(re^{i\theta}) = (P_r * K_\rho * h)(e^{i\theta}), \quad h \in H^2(T),$$

is 1-1 from $H^2(T)$ onto D_ρ . Further, the norm $\| \cdot \|_\rho^$ of (1.5) induced on $T_\rho h \in D_\rho$ by taking the $H^2(T)$ -norm of h is equivalent to the norm $\| \cdot \|_\rho$ in (1.1).*

PROOF. The assertions concerning K_ρ follow from Lemma 2.1 once it is observed $c_n = [n\rho(1/n)]^{-1/2}$, $n \geq 1$, is nonincreasing and convex as a consequence of the concavity of ρ .

We next prove $\| \cdot \|_\rho$ and $\| \cdot \|_\rho^*$ are equivalent, which implies $D_\rho \subset T_\rho(H^2(T))$, as was pointed out in §1. Now, Plancherel's theorem shows that for $f(z) = \sum_{n=0}^\infty a_n z^n$ in $H(U)$,

$$\begin{aligned} \iint_U |f'(z)|^2 \rho(1 - |z|) dz &= \int_0^1 \rho(1 - r)r dr \int_0^{2\pi} \left| \sum_{n=1}^\infty n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 d\theta \\ &= 2\pi \sum_{n=1}^\infty n^2 |a_n|^2 \left[\int_0^1 r^{2n-1} \rho(1 - r) dr \right]. \end{aligned}$$

We must prove

$$\int_0^1 r^{2n-1} \rho(1 - r) dr = \int_0^1 \rho(r)(1 - r)^{2n-1} dr \approx n^{-1} \rho(1/n).$$

Since ρ is increasing,

$$\begin{aligned} \left[\frac{(1 - 1/n)^{2n}}{2} \right] n^{-1} \rho(1/n) &= \rho(1/n) \int_{1/n}^1 (1 - r)^{2n-1} dr \\ &\leq \int_{1/n}^1 \rho(r)(1 - r)^{2n-1} dr \leq \int_0^1 \rho(r)(1 - r)^{2n-1} dr \end{aligned}$$

and

$$\int_0^{1/n} \rho(r)(1 - r)^{2n-1} dr \leq \rho(1/n) \int_0^{1/n} (1 - r)^{2n-1} dr \leq n^{-1} \rho(1/n).$$

Moreover, $\rho(r)/r$ nonincreasing implies

$$\int_{1/n}^1 \rho(r)(1 - r)^{2n-1} dr \leq n\rho(1/n) \int_{1/n}^1 r(1 - r)^{2n-1} dr.$$

But, integration by parts, followed by elementary estimates, yields

$$\int_{1/n}^1 r(1 - r)^{2n-1} dr \leq n^{-2}.$$

This completes the proof of the equivalence of the norms.

To see T_ρ is 1-1 onto D_ρ we note that as

$$(T_\rho h)(re^{i\theta}) = (P_r * K_\rho * h)(e^{i\theta}) = \sum_{n=-\infty}^\infty a_n r^n e^{in\theta},$$

then

$$[|n|\rho(1/|n|)]^{1/2} a_n = [|n|\rho(1/|n|)]^{1/2} \hat{K}_\rho(n) \hat{h}(n) = \begin{cases} \hat{h}(n), & n > 0, \\ 0, & n < 0. \end{cases}$$

III. Carleson measures. In this section we obtain a characterization of the Carleson measures of S_ρ and then use it to complete the proof of Theorem A.

Let K be any even, nonnegative function in $L^1(T)$. We wish to determine those $\mu \in B(U)$ for which the operator $T_K: h \rightarrow P_r * K * h$ is bounded from $L^2(T)$ to $L^2(\mu)$. Indeed, we consider the more general problem in which the index 2 is

replaced by any fixed $p, 1 < p < \infty$. Such a measure μ will be called a Carleson measure on

$$S_K^p = \{P_r * K * h : h \in L^p(T)\}.$$

It turns out to be easier to deal with the equivalent problem for the dual operator T'_K defined at $\nu \in B(U)$ by

$$(T'_K \nu)(e^{i\theta}) = \iint_U (P_r * K)(e^{i(\theta-\phi)}) d\nu(re^{i\phi}).$$

Our result, Theorem 3.1, is given in terms of the maximal operator M_K which sends a positive Borel measure ν on U to a function $M_K \nu$ on T , with

$$(M_K \nu)(e^{i\theta}) = \sup \left[|I|^{-1} \int_0^{|I|} K(e^{i\phi}) d\phi \right] \int_{S(I)} d\nu,$$

the supremum being over all subarcs I of T containing $e^{i\theta}$. For K even and non-negative on T , let $\tilde{K}(e^{i\theta}) = \sup_{|\theta| \leq \phi \leq \pi} K(e^{i\phi})$ denote the least even nonincreasing majorant of K . For $1 < p < \infty, p' = p/(p-1)$.

THEOREM 3.1. *Fix $p \in (1, \infty)$ and suppose K is an even, nonnegative function in $L^1(T)$ such that*

$$(3.1) \quad \int_0^\theta \tilde{K}(e^{i\phi}) d\phi \leq C \int_0^\theta K(e^{i\phi}) d\phi, \quad 0 < \theta < \pi.$$

Finally, let $\mu \in B(U)$. Then $T_K : L^p(T) \rightarrow L^p(\mu)$ if and only if $C > 0$ exists such that

$$(3.2) \quad \int_I [M_K(\chi_{S(I)} \mu)(e^{i\theta})]^{p'} d\phi \leq C \int_{S(I)} d\mu < \infty$$

for subarcs I of T .

The proof of Theorem 3.1 follows the line of argument used by the authors in [4] (see also [3]) to study the L^p trace inequality for convolution operators with radially decreasing kernels. The first step is to relate T'_K and M_K (cf. Bonami-Johnson [1]).

THEOREM 3.2. *Let p, K and \tilde{K} be as in Theorem 3.1. Then,*

(a) *There is $C > 0$ such that $(M_K \nu)(e^{i\theta}) \leq CM(T'_K \nu)(e^{i\theta}), e^{i\theta} \in T$, for all $\nu \in B(U)$. Here M is the classical Hardy-Littlewood maximal operator on T ,*

$$(Mg)(e^{i\theta}) = \sup_{e^{i\theta} \in I} |I|^{-1} \int_I |g(e^{i\phi})| d\phi.$$

(b) *There exists $\gamma > 1$ and $C' > 0$ such that for all $\lambda > 0$, all $\beta \in (0, 1]$ and all $\nu \in B(U)$,*

$$\begin{aligned} & |\{e^{i\theta} \in T : (T'_K \nu)(e^{i\theta}) > \gamma\lambda, (M_K \nu)(e^{i\theta}) \leq \beta\lambda\}| \\ & \leq C' \beta |\{e^{i\theta} \in T : M(T'_K \nu)(e^{i\theta}) > \lambda\}|. \end{aligned}$$

PROOF. (a) Fix a subarc I of T . Then,

$$\begin{aligned}
 \int_I (T'_K \nu)(e^{i\theta}) d\theta &= \int_I d\theta \int \int_U (P_r * K)(e^{i(\theta-\phi)}) d\nu(re^{i\phi}) \\
 (3.3) \qquad &= \int \int_U d\nu(re^{i\phi}) \int_I (P_r * K)(e^{i(\theta-\phi)}) d\theta \\
 &\geq \left[\inf_{re^{i\phi} \in S(I)} \int_I (P_r * K)(e^{i(\theta-\phi)}) d\theta \right] \int_{S(I)} d\nu(re^{i\phi}).
 \end{aligned}$$

However, for $re^{i\phi} \in S(I)$,

$$\begin{aligned}
 \int_I (P_r * K)(e^{i(\theta-\phi)}) d\theta &= \int_I d\theta \int_T P_r(e^{i(\theta-\phi-t)}) K(e^{it}) dt \\
 (3.4) \qquad &= \int_T K(e^{it}) dt \int_I P_r(e^{i(\theta-\phi-t)}) d\theta \\
 &\geq \int_{-|I|}^{|I|} K(e^{it}) dt \int_I P_r(e^{i(\theta-\phi-t)}) d\theta \\
 &\geq C \int_0^{|I|} K(e^{it}) dt,
 \end{aligned}$$

since

$$\int_I P_r(e^{i(\theta-\phi-t)}) d\theta \geq \int_{|I|}^{2|I|} P_r(e^{is}) ds \geq C,$$

when $(1 - |I|)_+ \leq r < 1$ and $\phi + t \in 3I$. Combining (3.3) and (3.4) yields

$$(3.5) \qquad |I|^{-1} \int_I (T'_K \nu)(e^{i\theta}) d\theta \geq C \left[|I|^{-1} \int_0^{|I|} K(e^{it}) dt \right] \int_{S(I)} d\nu(re^{i\theta})$$

for all I . Now fix $e^{i\theta} \in T$ and take the supremum in (3.5) over all I containing $e^{i\theta}$ to complete the proof of (a).

(b) Let $\lambda > 0$ be given and set

$$\Omega_\lambda = \{e^{i\theta} \in T: M(T_{\tilde{K}} \nu)(e^{i\theta}) > \lambda\}.$$

Let $\{I_K\}$ be those component subarcs of Ω_λ for which $e^{i\theta_k} \in I_k$ exists such that $(M_K)(e^{i\theta_k}) \leq \beta\lambda$. Fix such a subarc and denote it by I . Let $3I$ be the subarc with the same centre as I but 3 times the length. We have

$$(3.6) \qquad (3|I|)^{-1} \left[\int_0^{3|I|} K(e^{i\theta}) d\theta \right] \left[\int_{S(3I)} d\nu(re^{i\phi}) \right] \leq \beta\lambda;$$

$$(3.7) \qquad |I|^{-1} \int_I (T'_{\tilde{K}} \nu)(e^{i\theta}) d\theta \leq \lambda,$$

since $(M_K \nu)(e^{i\theta}) \leq \beta\lambda$ for some $e^{i\theta} \in I$ and since $M(T'_{\tilde{K}} \nu) \leq \lambda$ at each end of I . Define $\nu_1 = \nu|_{S(3I)}$ and $\nu_2 = \nu - \nu_1 = \nu|_{S(3I)^c}$. It will be sufficient to obtain

$$(3.8) \qquad (T'_{\tilde{K}} \nu_2)(e^{i\theta}) \leq C_1 \lambda, \qquad e^{i\theta} \in I,$$

for some $C_1 > 0$ independent of I . To see this, suppose (3.8) holds and $\gamma > 2C_1$. Then,

$$\begin{aligned}
 (3.9) \quad & |\{e^{i\theta} \in I: (T'_{\tilde{K}}\nu)(e^{i\theta}) > \gamma\lambda\}| \\
 & \leq |\{e^{i\theta} \in I: (T'_{\tilde{K}}\nu_1)(e^{i\theta}) > C_1\lambda\}| \\
 & \leq (C_1\lambda)^{-1} \int_I (T'_{\tilde{K}}\nu_1)(e^{i\theta}) d\theta.
 \end{aligned}$$

Now

$$\begin{aligned}
 (3.10) \quad & \int_I (T'_{\tilde{K}}\nu_1)(e^{i\theta}) d\theta = \int_I d\theta \iint_{S(3I)} (P_r * \tilde{K})(e^{i(\theta-\phi)}) d\nu(re^{i\phi}) \\
 & = \iint_{S(3I)} d\nu(re^{i\phi}) \int_I (P_r * \tilde{K})(e^{i(\theta-\phi)}) d\theta
 \end{aligned}$$

and

$$\begin{aligned}
 (3.11) \quad & \int_I (P_r * \tilde{K})(e^{i(\theta-\phi)}) d\theta = \int_I d\theta \int_T P_r(e^{it}) \tilde{K}(e^{i(\theta-\phi-t)}) dt \\
 & = \int_T P_r(e^{it}) dt \int_I \tilde{K}(e^{i(\theta-\phi-t)}) d\theta \\
 & \leq \int_T P_r(e^{it}) dt \int_{-|I|/2}^{|I|/2} \tilde{K}(e^{is}) ds \leq 2 \int_0^{|I|} \tilde{K}(e^{is}) ds,
 \end{aligned}$$

since $\tilde{K}(e^{is})$ is even when $s \in [-\pi, \pi]$ and nonincreasing when $s \in [0, \pi]$. Combining (3.9), (3.10) and (3.11) we obtain

$$\begin{aligned}
 (3.12) \quad & |\{e^{i\theta} \in I: (T'_{\tilde{K}}\nu)(e^{i\theta}) > \gamma\lambda\}| \\
 & \leq 2(C_1\lambda)^{-1} \left[\int_0^{3|I|} \tilde{K}(e^{is}) ds \right] \left[\iint_{S(3I)} d\nu(re^{i\phi}) \right] \\
 & \leq CC_1^{-1}\beta|I|,
 \end{aligned}$$

where the last inequality follows from (3.1) and (3.6). Summing (3.12) over all the I_k gives (b).

It remains to prove (3.8). We claim this follows from the fact that

$$(3.13) \quad (P_r * \tilde{K})(e^{i(\theta-\phi)}) \leq C|I|^{-1} \int_{|\theta-\phi-t| \leq |I|} (P_r * \tilde{K})(e^{it}) dt$$

whenever $e^{i\theta} \in I$ and $re^{i\phi} \notin S(3I)$. For given (3.13), we have

$$\begin{aligned}
 (T'_{\tilde{K}}\nu_2)(e^{i\theta}) &= \iint_{S(3I)^c} (P_r * \tilde{K})(e^{i(\theta-\phi)}) d\nu(re^{i\phi}) \\
 &\leq C|I|^{-1} \iint_{S(3I)^c} d\nu(re^{i\phi}) \int_{|\theta-\phi-t| \leq |I|} (P_r * \tilde{K})(e^{it}) dt \\
 &\leq C|I|^{-1} \iint_U d\nu(re^{i\phi}) \int_{|s| \leq |I|} (P_r * \tilde{K})(e^{i(\theta-\phi-s)}) ds \\
 &= C|I|^{-1} \int_{|s| \leq |I|} (T'_{\tilde{K}}\nu)(e^{i(\theta-s)}) ds \\
 &\leq C\lambda \quad \text{by (3.7),}
 \end{aligned}$$

as required by (3.8).

To see (3.13) observe that if $re^{i\phi} \notin S(3I)$, then either $e^{i\phi} \notin 3I$, in which case $|I| \leq |\theta - \phi|$ since $e^{i\theta} \in I$; or $0 \leq r < 1 - 3|I|$. In the former case (3.13) holds, since $(P_r * \tilde{K})(e^{is})$ is even for $s \in [-\pi, \pi]$ and nonincreasing for $s \in (0, \pi]$. In the latter case we use the inequality

$$P_r(e^{it}) \leq C|I|^{-1} \int_{|s| \leq |I|} P_r(e^{i(t-s)}) ds, \quad -\pi \leq t \leq \pi, \quad 0 < r < 1 - 3|I|.$$

(For the case $|t| \leq |I|$, use the estimate $P_r(e^{iu}) \approx 1/(1-r)$, $|u| \leq 2|I|$; for the case $|t| > |I|$, use the fact that $u \rightarrow P_r(e^{iu})$ is decreasing away from 0.) Thus the left side of (3.13),

$$\int_T P_r(e^{i(\theta-\phi-u)}) \tilde{K}(e^{iu}) du,$$

is dominated by

$$\begin{aligned} C \int_T |I|^{-1} \int_{|s| \leq |I|} P_r(e^{i(\theta-\phi-u-s)}) ds \tilde{K}(e^{iu}) du \\ = C|I|^{-1} \int_{|\theta-\phi-t| \leq |I|} dt \int_T P_r(e^{i(t-u)}) \tilde{K}(e^{iu}) du \end{aligned}$$

which equals the right side of (3.13).

The second step in the proof of Theorem 3.1 involves the following analogue of a two-weight norm inequality for maximal operators in [7]; the proof is a straightforward adaptation of ones given in [7] and so is omitted.

THEOREM 3.3. *Let K be as in Theorem 3.1 and let $q \in (1, \infty)$. Suppose $\mu \in B(U)$, $\nu \in B(T)$. Then, the inequality*

$$\int_T [M_K(f\mu)(e^{i\theta})]^q d\nu(e^{i\theta}) \leq C \iint_U f(re^{i\phi})^q d\nu(re^{i\phi})$$

holds for all $f \geq 0$ on U if and only if

$$\int_I [M_K(\chi_{S(I)}\mu)(e^{i\theta})]^q d\nu(e^{i\theta}) \leq C \iint_{S(I)} d\mu(re^{i\phi})$$

for all subarcs I of T .

We are now ready to give the

PROOF OF THEOREM 3.1. By duality, $T_K: L^P(T) \rightarrow L^P(\mu)$ if and only if

$$(3.14) \quad \int_T [T'_K(f\mu)(e^{i\theta})]^{p'} d\theta \leq C \iint_U f(re^{i\phi})^{p'} d\mu(re^{i\phi})$$

for all $f \geq 0$ in $L^{p'}(\mu)$. Part (a) of Theorem 3.2, together with the Hardy-Littlewood maximal theorem (see [9, p. 5]), shows that

$$(3.15) \quad \begin{aligned} \int_T [M_K(f\mu)(e^{i\theta})]^{p'} d\theta &\leq C \int_T [M(T'_K(f\mu))(e^{i\theta})]^{p'} d\theta \\ &\leq C \int_T [T'_K(f\mu)(e^{i\theta})]^{p'} d\theta. \end{aligned}$$

Again, by part (b) of Theorem 3.2 and the maximal theorem in [9],

$$\begin{aligned}
 & \int_T [T'_{\tilde{K}}(f\mu)(e^{i\theta})]^{p'} d\theta \\
 &= p' \gamma^{p'} \int_0^\infty \lambda^{p'-1} |\{\theta: T'_{\tilde{K}}(f\mu)(e^{i\theta}) > \gamma\lambda\}| d\lambda \\
 (3.16) \quad &\leq C \int_0^\infty \lambda^{p'-1} |\{M_K(f\mu)(e^{i\theta}) > \beta\lambda\}| d\lambda \\
 &\quad + C\beta \int_0^\infty \lambda^{p'-1} |\{M(T'_{\tilde{K}}f\mu)(e^{i\theta}) > \lambda\}| d\lambda \\
 &\leq C\beta^{-p'} \int_T [M_K(f\mu)(e^{i\theta})]^{p'} d\theta + C\beta \int_T [M(T'_{\tilde{K}}(f\mu))(e^{i\theta})]^{p'} d\theta \\
 &\leq C\beta^{-p'} \int_T [M_K(f\mu)(e^{i\theta})]^{p'} d\theta + C\beta \int_T [(T'_{\tilde{K}}(f\mu))(e^{i\theta})]^{p'} d\theta.
 \end{aligned}$$

Choosing β so small that $C\beta < 1/2$ and subtracting $C\beta \int_T [T'_{\tilde{K}}(f\mu)(e^{i\theta})]^{p'} d\theta$ from both sides of (3.16) yields

$$\begin{aligned}
 (3.17) \quad & \int_T [T'_K(f\mu)(e^{i\theta})]^{p'} d\theta \leq \int_T [T'_{\tilde{K}}(f\mu)(e^{i\theta})]^{p'} d\theta \\
 & \leq C \int_T [M_K(f\mu)(e^{i\theta})]^{p'} d\theta.
 \end{aligned}$$

From (3.15) and (3.17) we conclude that (3.14) holds if and only if

$$\int_T [M_K(f\mu)(e^{i\theta})]^{p'} d\theta \leq C \iint_U f(re^{i\phi})^{p'} d\mu(re^{i\phi})$$

for all $f \geq 0$. Finally, Theorem 3.3 with $d\nu(e^{i\theta}) = d\theta$ yields the conclusion of Theorem 3.1.

As pointed out in §1, $g \in M(D_\rho)$ if and only if $g \in H^\infty(U)$ and $T_{K_\rho} : L^2(T) \rightarrow L^2(\mu)$, where $d\mu(z) = |g'(z)|^2 \rho(1 - |z|) dz$. We now know the boundedness of T_{K_ρ} is equivalent to (3.2) holding for $K = K_\rho$ and $p = 2$, provided it can be shown that K_ρ satisfies (3.1). We complete the proof of Theorem A by showing (3.1) for $K = K_\rho$, as well as the equivalence of M_{K_ρ} and M_ρ in

LEMMA 3.4. *Suppose ρ is a Dirichlet weight and let $K = K_\rho$ be defined by (1.6). If \tilde{K} denotes the least nonincreasing even majorant of K , then one has the equivalences*

$$x^{-1} \int_0^x K(y) dy \approx x^{-1} \int_0^x \tilde{K}(y) dy \approx [x\rho(x)]^{-1/2}, \quad x \in [0, \pi].$$

PROOF. By the definition of \tilde{K} , $x^{-1} \int_0^x K(y) dy \leq x^{-1} \int_0^x \tilde{K}(y) dy$. Summing by parts we obtain $K(x) = \sum_{n=0}^\infty (n+1)\Delta^2 \hat{K}(n)\phi_n(x)$, where

$$\phi_n(x) = (n+1)^{-1} \left[\frac{\sin[(n+1)x/2]}{\sin(x/2)} \right]^2$$

is the Féjer kernel of order n . Thus, recalling that $\{\hat{K}(n)\}$ is convex, we obtain

$$K(y) \leq \sum_{n=0}^\infty (n+1)\Delta^2 \hat{K}(n) \sup_{|y| \leq z \leq \pi} \phi_n(z).$$

But, it is easily seen that

$$\int_0^x \left[\sup_{|y| \leq z \leq \pi} \phi_n(z) \right] dy \leq C \int_0^x \phi_n(y) dy$$

for $C > 0$ independent of n . Hence, the averages of K and \tilde{K} are equivalent.

With $N = [1/x]$,

$$(3.18) \quad x^{-1} \int_0^x K(y) dy \approx \int_{-\pi}^{\pi} \phi_N(y) K(y) dy \approx \sum_{|n| \leq [1/x]} \left(1 - \frac{|n|}{N+1} \right) \hat{K}(n);$$

the last equivalence is a consequence of the general form of Parseval's formula. To complete the proof of the lemma it will be sufficient to prove $x^{-1} \int_0^x K(y) dy \approx [x\rho(x)]^{-1/2}$ for x near 0, say $x \in (0, 1/4)$. For such x , we have from (1.6) and (3.18)

$$\begin{aligned} C^{-1} \sum_{n=1}^{[1/x]/2} [n\rho(1/n)]^{-1/2} &\leq x^{-1} \int_0^x K(y) dy \\ &\leq C \sum_{n=1}^{[1/x]} [n\rho(1/n)]^{-1/2}, \end{aligned}$$

or, equivalently (since ρ concave implies $\rho(z)/z$ nonincreasing)

$$\begin{aligned} C^{-1} \int_1^{1/2x} [y\rho(1/y)]^{-1/2} dy &\leq x^{-1} \int_0^x K(y) dy \\ &\leq C \int_1^{1/x} [y\rho(1/y)]^{-1/2} dy. \end{aligned}$$

But, since $\rho(z)/z$ nonincreasing,

$$\begin{aligned} \int_1^{1/2x} [y\rho(1/y)]^{-1/2} dy &\geq [\rho(2x)/2x]^{-1/2} \left(\frac{1}{2x} - 1 \right) \\ &\geq 4^{-1} [x\rho(x)]^{-1/2}, \end{aligned}$$

while

$$\int_1^{1/x} [y\rho(1/y)]^{-1/2} dy \leq \rho(x)^{-1/2} \int_1^{1/x} y^{-1/2} dy \leq 2[x\rho(x)]^{-1/2}.$$

IV. The Féjer-Riesz inequality. Finally, we specialize Theorem 3.3 to the case in which $K = K_\rho$, so that M_{K_ρ} is equivalent to M_ρ ; ν is Lebesgue measure on T ; μ is carried by the line segment $L = \{z \in U: \text{Im } z = 0, 0 \leq \text{Re } z < 1\}$.

COROLLARY 4.5. *Suppose ρ is a Dirichlet weight and let $\mu \in B(U)$ be carried by L . Then, μ is a Carleson measure on S_ρ if and only if*

$$(4.1) \quad \int_0^t \left[\sup_{x \leq s \leq t} [s\rho(s)]^{-1/2} \int_{1-s}^1 d\mu \right]^2 dx \leq C \int_{1-t}^1 d\mu < \infty,$$

whenever $0 < t < 1$.

Condition (4.1) suggests a natural way to construct a Carleson measure for S_ρ which is absolutely continuous with respect to Lebesgue measure on L . The idea

is to suppose equality holds in (4.1) and that the supremum in square brackets on the left side is attained at $s = x$. With $F(t) = \int_{1-t}^1 d\mu$, this means

$$\int_0^t [x\rho(x)]^{-1} F(x)^2 dx = CF(t), \quad 0 < t < 1.$$

If we further normalize μ so that $F(1) = 1$ and set $C = 1$, then F satisfies the boundary value problem

$$\begin{cases} F'(t) = [t\rho(t)]^{-1} F(t)^2, \\ F(0+) = 0, \quad F(1) = 1 \end{cases}$$

whose solution is

$$F(t) = \left[1 + \int_t^1 [s\rho(s)]^{-1} ds \right]^{-1}.$$

Let

$$(4.2) \quad d\mu(t) = dF(t) = [t\rho(t)]^{-1} \left[1 + \int_t^1 [s\rho(s)]^{-1} ds \right]^{-2} dt$$

on L . Then (4.1) will hold provided the supremum in square brackets on the left side is attained when $s = x$; that is, provided the function

$$G(y) = [y\rho(y)]^{-1/2} F(y) = [y\rho(y)]^{-1/2} \left[1 + \int_y^1 [s\rho(s)]^{-1} ds \right]^{-1}$$

is nonincreasing. While this is not the case, it is true that G is almost decreasing; that is,

$$(4.3) \quad G(y) \leq 2G(x), \quad 0 < x \leq y < 1,$$

and this is enough to force (4.1). Since $G(1) = 1$ and, by L'Hôpital's rule,

$$\begin{aligned} \lim_{y \rightarrow 0+} G(y) &= \lim_{y \rightarrow 0+} [\rho(y) + y\rho'(y)] / 2[y\rho(y)]^{1/2} \\ &\geq \lim_{y \rightarrow 0+} 2^{-1} [\rho(y)/y]^{1/2} = \infty, \end{aligned}$$

it suffices to prove (4.3) when both x and y are critical points of G or x is critical and $y = 1$. However, if $G'(z) = 0$, then

$$1 + \int_z^1 [t\rho(t)]^{-1} dt = 2/[\rho(z) + z\rho'(z)].$$

So, for x and y critical,

$$\begin{aligned} G(y) &= 2^{-1} [\rho(y) + y\rho'(y)] / [y\rho(y)]^{1/2} \\ &\leq [\rho(y)/y]^{1/2} \quad (y\rho'(y) \leq \rho(y)) \\ &\leq [\rho(x)/x]^{1/2} \quad (x \leq y) \\ &\leq [\rho(x) + x\rho'(x)] / [x\rho(x)]^{1/2} \quad (x\rho'(x) \geq 0) \\ &= 2G(x), \end{aligned}$$

and for x critical and $y = 1$,

$$G(y) = 1 \leq [p(x)/x]^{1/2} \leq 2G(x).$$

To summarize, we have proved

COROLLARY 4.6 (CF. NAGEL, RUDIN AND SHAPIRO [6]). *Suppose ρ is a Dirichlet weight and let μ be the measure carried by L given by (4.2). Then μ is a Carleson measure for S_ρ .*

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